

Morris-Thorne wormholes in static pseudo-spherically symmetric spacetimes

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In this paper we study classical general relativistic static wormhole configurations with pseudo-spherical symmetry. We show that in addition to the hyperbolic wormhole solutions discussed by Lobo and Mimoso in the Ref. Phys. Rev. D **82**, 044034 (2010), there exists another wormhole class, which is truly pseudo-spherical counterpart of spherical Morris-Thorne wormhole (contrary to the Lobo-Mimoso wormhole class), since all constraints originally defined by Morris and Thorne for spherically symmetric wormholes are satisfied. We show that, for both classes of hyperbolic wormholes the energy density, at the throat, is always negative, while the radial pressure is positive, contrary to the spherically symmetric Morris-Thorne wormhole. Specific hyperbolic wormholes are constructed and discussed by imposing different conditions for the radial and lateral pressures, or by considering restricted choices for the redshift and the shape functions. In particular, we show that an hyperbolic wormhole can not be sustained at the throat by phantom energy, and that there are pseudo-spherically symmetric wormholes supported by matter with isotropic pressure and characterized by space sections with an angle deficit (or excess).

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I. INTRODUCTION

In classical general relativity the Einstein field equations admit a simple and interesting class of static solutions describing tunnels in spacetime, connecting either two remote regions of our Universe or even different universes. Far from the tunnel, spacetime may either be flat or curved geometry. These geometrical configurations are called wormholes.

The spacetime wormhole ansatz of Morris and Thorne was formulated originally for static spherically symmetric metrics in the form [1, 2]

$$ds^2 = e^{2\phi(r)} dt^2 - \frac{dr^2}{1 - \frac{b(r)}{r}} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where $e^{\phi(r)}$ and $b(r)$ are arbitrary functions of the radial coordinate. In general, any static spherically symmetric spacetime may be written in the form of Eq. (1). In the case of wormhole configurations $e^{\phi(r)}$ and $b(r)$ are referred to as redshift function and shape function respec-

tively. In order to have a wormhole these two functions must satisfy some general constraints discussed by the authors in Ref. [1, 2]. These constraints provide a minimum set of conditions which lead to a geometry featuring two regions connected by a bridge, and are defined by:

Constraint 1: A no-horizon condition, i.e. $e^{\phi(r)}$ is finite throughout the spacetime in order to ensure the absence of horizons and singularities.

Constraint 2: Minimum value of the r -coordinate, i.e. at the throat of the wormhole $r = b(r) = r_0$, r_0 being the minimum value of r .

Constraint 3: Finiteness of the proper radial distance, i.e.

$$\frac{b(r)}{r} \leq 1, \quad (2)$$

(for $r \geq r_0$) throughout the spacetime. The equality sign holds only at the throat. Eq. (2) is required in order to ensure the finiteness of the proper radial distance $l(r)$ defined by

$$l(r) = \pm \int_{r_0}^r \frac{dr}{\sqrt{1 - \frac{b(r)}{r}}}, \quad (3)$$

where the \pm signs refer to the two asymptotically flat regions which are connected by the wormhole. Note that

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the condition (2) assures that the metric component g_{rr} in Eq. (1) does not change its sign for any $r \geq r_0$.

Constraint 4: Asymptotic flatness condition, i.e. as $l \rightarrow \pm\infty$ (or equivalently, $r \rightarrow \infty$) then

$$\frac{b(r)}{r} \rightarrow 0. \quad (4)$$

Although asymptotically flat wormhole geometries are of particular interest more general wormhole spacetimes also have been studied in the literature [3].

Wormholes are sustained by exotic matter sources which would violate all the energy conditions [1, 2]. Static wormholes supported by phantom energy also have been constructed [4, 5]. In this case the notion of phantom energy is used in a more general sense than it has been used in standard cosmology, where sources are characterized by an isotropic pressure. For static spacetimes the phantom matter threading the wormholes is essentially an inhomogeneous and anisotropic fluid with radial and lateral pressures satisfying the relation $p_r \neq p_l$. This inhomogeneous and anisotropic phantom matter has a very strong negative radial pressure with the equation of state $p_r/\rho < -1$ (note that this implies that the energy density is positive).

It is remarkable that wormhole configurations may be constructed without needing any form of exotic matter sources in the framework of alternative theories of gravity, such as Einstein-Gauss-Bonnet theory, Lovelock models, Brans-Dicke theory, among others [6].

In this paper we examine the extension of spherically symmetric Morris and Thorne wormholes to pseudo-spherically symmetric Lorentzian spacetimes, i.e. to wormholes with negatively curved spatial sections satisfying the four constraints enumerated above.

The first analysis of static pseudo-spherically symmetric wormholes was made in Ref. [7], where the authors explored physical properties and characteristics of these hyperbolic solutions, and analyzed some specific static wormhole solutions supported by exotic matter. Lobo and Mimoso study a class of hyperbolic wormholes which differ radically from the Morris-Thorne wormholes. In particular, this type of wormholes does not satisfy the condition (2), on the other hand, for example, it was shown that at the throat the energy density of the material threading the hyperbolic spacetime tunnel is always negative, while the radial pressure is positive, contrary to the spherically symmetric Morris-Thorne counterpart.

In this work we further explore these pseudo-spherically symmetric wormholes. In particular, we show that there exist another class of hyperbolic wormholes that satisfy all constraints summarized and enumerated above, which were originally formulated by Morris and Thorne (in the following M-T).

In order to compare both classes of pseudo-spherical wormholes we summarize the constraints of traversable hyperbolic tunnels formulated by Lobo and Mimoso (in the following L-M). The spacetime wormhole ansatz of

Lobo and Mimoso is written in the form [7]

$$ds^2 = e^{2\phi(r)} dt^2 - \frac{dr^2}{\frac{b(r)}{r} - 1} - r^2 (d\theta^2 + \sinh^2 \theta d\phi^2), \quad (5)$$

where $e^{\phi(r)}$ and $b(r)$ are the redshift and the shape functions respectively.

The Constraint 1, on the absence of horizons, and Constraint 2, on the minimum value of the r -coordinate at the throat of the wormhole, are the same, implying that we must require that $e^{\phi(r)}$ is finite throughout the spacetime, and that $b(r_0) = r_0$, respectively.

For the Constraint 3 the Eq. (2) is not more fulfilled by hyperbolic L-M wormholes. The metric (5) imposes the condition

$$\frac{b(r)}{r} \geq 1, \quad (6)$$

for having $g_{rr} > 0$ for $r \geq r_0$.

The current formulation of the M-T Constraint 4, on asymptotic flatness condition, clearly is not fulfilled by the metric (5), since Eq. (6) implies that, as $r \rightarrow \infty$, the relation (4) can not be fulfilled at all (note also that if $b(r)/r \rightarrow 0$, then the radial metric component in Eq. (5) becomes negative).

Notice that Lobo and Mimoso consider that the constructed static and pseudo-spherically symmetric spacetime tunnels are made by adding exotic matter to the vacuum solution $e^{\phi(r)} = 2\mu/r - 1$, $b(r) = 2\mu$, which is the pseudo-spherically counterpart of the vacuum Schwarzschild solution. We shall address this statement in the conclusion section.

The paper is organized as follows. In Sec. II we write the Einstein equations for static pseudo-spherically symmetric spacetimes. In Sec. III we analyze hyperbolic wormholes sustained by a matter source with anisotropic pressure, characterized by a linear equation of state for the radial pressure, and obtain hyperbolic counterparts of spherical zero-tidal-force wormholes. In Sec. IV we analyze pseudo-spherical wormhole models sustained by a fluid with isotropic pressure. In Section V we discuss our results.

II. FIELD EQUATIONS FOR STATIC PSEUDO-SPHERICALLY SYMMETRIC SPACETIMES

In order to describe hyperbolic M-T wormhole class we shall use the wormhole ansatz in the form

$$ds^2 = e^{2\phi(r)} dt^2 - \frac{dr^2}{1 - \frac{b(r)}{r}} - r^2 d\Omega_{ps}^2, \quad (7)$$

where $d\Omega_{ps}^2 = d\theta^2 + \sinh^2 \theta d\phi^2$, $e^{\phi(r)}$ and $b(r)$ are arbitrary functions of the radial coordinate, and the usual two-dimensional spheres $d\theta^2 + \sin^2 \theta d\phi^2$ of the metric (1) are replaced by two-dimensional pseudo-spheres

$d\theta^2 + \sinh^2 \theta d\phi^2$. The coordinates θ and ϕ in this case are defined in the ranges $-\infty < \theta < \infty$ and $0 \leq \phi \leq 2\pi$ respectively. In general, any static pseudo-spherically symmetric spacetime may be written in the form (5) or (7). The essential characteristics of a wormhole geometry are still encoded in the spacelike section, so as in spherically symmetric wormholes, we shall call $e^{\phi(r)}$ and $b(r)$ the redshift and shape functions respectively.

The L-M metric (5) differs from Eq. (7) only by the metric component g_{rr} . It becomes clear that if we impose conditions on the redshift function or on the energy density and pressures (by requiring, for example, an equation of state) the spacetimes generated by the metric ansatz (5) and (7), and satisfying the Einstein equations, will be the same solutions. However, if we construct wormhole solutions by imposing specific restricted choices for the shape function, the constraints imposed by the metrics (5) and (7) on $b(r)$ may be quite different. Effectively, if we consider for example the shape function in the form $b(r) = r_0(r/r_0)^\alpha$, the ansatz metric (5), used in Ref. [7], implies that the radial metric component is given by $g_{rr}^{-1} = \left(\frac{r}{r_0}\right)^{\alpha-1} - 1$. Then, in order to have $g_{rr} > 0$, for $r \geq r_0$, we must require that $\alpha > 1$. On the other hand, for the same form of the shape function we have that the metric (7) implies that the radial metric component is given by $g_{rr}^{-1} = 1 - \left(\frac{r}{r_0}\right)^{\alpha-1}$. Then, in order to have $g_{rr} > 0$, for $r \geq r_0$, we must require that $\alpha < 1$. This statement will be addressed below by considering this specific form of the shape function $b(r)$.

By assuming that the matter content is described by a single imperfect fluid, from the Einstein field equations we obtain

$$\kappa\rho(r) = \frac{b' - 2}{r^2}, \quad (8)$$

$$\kappa p_r(r) = 2 \left(1 - \frac{b}{r}\right) \frac{\phi'}{r} - \frac{b}{r^3} + \frac{2}{r^2}, \quad (9)$$

$$\kappa p_l(r) = \left(1 - \frac{b}{r}\right) \times \left[\phi'' + \phi'^2 - \frac{b'r + b - 2r}{2r(r-b)} \phi' - \frac{b'r - b}{2r^2(r-b)} \right], \quad (10)$$

where $\kappa = 8\pi G$, prime denotes $\partial/\partial r$, ρ is the energy density, and p_r and p_l are the radial and lateral pressures respectively.

From Eqs. (8)-(10) we obtain the hydrostatic equation for equilibrium of the matter content sustaining the hyperbolic wormhole geometry, which is given by

$$p_r' = \frac{2(p_l - p_r)}{r} - (\rho + p_r)\phi'. \quad (11)$$

Summarizing, we have a system of three equations (8)-(10), with five unknown functions: $e^{\phi(r)}$, $b(r)$, $\rho(r)$, $p_r(r)$, and $p_l(r)$. To construct hyperbolic wormholes we must constraint two of these five functions. In order to make this we will consider specific equations of state for p_r or

p_l , impose the condition $p_r = p_l$, for having isotropic pressure, or consider restricted choices for the redshift and shape functions.

III. HYPERBOLIC WORMHOLES WITH $p_r = \omega\rho$

Let us begin our study by considering pseudo-spherical wormhole solutions sustained by a matter with anisotropic pressure, characterized by the linear equation of state

$$p_r = \omega\rho, \quad (12)$$

for the radial pressure. By using Eqs. (8), (9) and (12) we obtain the following equation

$$\omega r b' + 2r(b-r)\phi' + b = 2(1+\omega)r. \quad (13)$$

For zero-tidal-force wormholes, i.e. $\phi(r) = 0$, we obtain

$$b(r) = 2r + Cr^{-1/\omega}, \quad (14)$$

where C is an integration constant, and then we may write the solution in the form

$$ds^2 = dt^2 - \frac{dr^2}{\left(\frac{r}{r_0}\right)^{-\frac{1+\omega}{\omega}} - 1} - r^2 d\Omega_{ps}^2, \quad (15)$$

$$\kappa\rho = \frac{\left(\frac{r}{r_0}\right)^{-(3\omega+1)/\omega}}{\omega r_0^2}, \quad (16)$$

$$\kappa p_l = -\frac{(1+\omega)\left(\frac{r}{r_0}\right)^{-(3\omega+1)/\omega}}{2\omega r_0^2}. \quad (17)$$

In this case we have that for $r \geq r_0$ the metric component $g_{rr}^{-1} \geq 0$ if $-1 \leq \omega < 0$. This implies that the energy density is negative, and the pressures p_r and p_l are positive. This solution was previously obtained by the authors of Ref. [7]. If we instead of the equation of state (12) require such a linear equation of state for the lateral pressure, i.e. $p_l = \tilde{\omega}\rho$, then the same solution is obtained. Both of them are connected by the transformation $\tilde{\omega} = -(1+\omega)/2$.

Now we will consider the wormhole solution generated by imposing on the metric (7) the shape function in the form $b(r) = r_0(r/r_0)^\alpha$. From Eq. (13) we obtain that

$$e^{\phi(r)} = \frac{A \left(r_0 \left(\frac{r}{r_0} \right)^\alpha - r \right)^{\frac{1+2\omega-\alpha\omega}{2(\alpha-1)}}}{\left(\frac{r}{r_0} \right)^{\frac{-1+\alpha(\omega+2)}{2(\alpha-1)}}}, \quad (18)$$

where A is an integration constant. Without any loss of generality we can put $A = 1$. Now, since Eq. (18) implies that $e^{\phi(r_0)} = 0$, an event horizon is located at r_0 , implying that in this case we can have only non-traversable hyperbolic wormholes (including the case $\alpha = 0$).

For avoiding this event horizon at r_0 , we need to impose that $\alpha = (1+2\omega)/\omega$, then $e^{\phi(r)} = \left(\frac{r}{r_0}\right)^{-\omega-1}$. Hence the solution takes the form

$$ds^2 = \left(\frac{r}{r_0}\right)^{-2(\omega+1)} dt^2 - \frac{dr^2}{1 - \left(\frac{r}{r_0}\right)^{\frac{\omega+1}{\omega}}} - r^2 d\Omega_{ps}^2, \quad (19)$$

$$\kappa\rho = \frac{(2\omega+1)\left(\frac{r}{r_0}\right)^{\frac{1+\omega}{\omega}} - 2\omega}{\omega r^2}, \quad (20)$$

$$\kappa p_l = -\frac{(1+\omega)\left(\left(\frac{r}{r_0}\right)^{\frac{1+\omega}{\omega}}(1+2\omega) - 2(1+\omega)\right)}{2r^2}, \quad (21)$$

where $\omega \neq -1$. In order to have a wormhole we must require $-1 < \omega < 0$. Note that at the throat the energy density (20) takes the value $\rho(r_0) = \frac{1}{r_0^2\omega}$, which is negative for $\omega < 0$. This matter configuration violates the strong energy condition since $\rho + p_r + 2p_l < 0$ at the throat. It is easy to see that for $-1 < \omega < 0$ and $r \geq r_0$ the energy density is always negative, and $p_r > 0$.

It is clear that the wormhole (15)-(17) is quite different from the wormhole (19)-(21). For the first one, we have that if $r \rightarrow \infty$ then $b(r)/r \rightarrow -\infty$, while for the second wormhole $b(r)/r \rightarrow 0$. The latter behavior is the same as of spherically symmetric M-T wormholes, since the Eq. (2) is satisfied. So we can say that the wormhole (19)-(21) is the hyperbolic version of the respective spherically symmetric wormhole

$$ds^2 = \left(\frac{r}{r_0}\right)^{-2(\omega+1)} dt^2 - \frac{dr^2}{1 - \left(\frac{r}{r_0}\right)^{\frac{\omega+1}{\omega}}} + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (22)$$

which may be obtained by the replacement of $\sinh\theta$ by $\sin\theta$. However, note that the spherical wormhole (22) does not fulfill the equation of state (12).

Lastly, in this section, we will construct the Lorentzian hyperbolic M-T wormhole by imposing to the metric (7) the restricted choices $b(r) = r_0(r/r_0)^\alpha$, for the shape function, and $\phi = 0$, for the redshift function. This wormhole is the counterpart of the zero-tidal-force spherically symmetric Morris and Thorne wormhole discussed in Ref. [1], which is given by

$$ds^2 = dt^2 - \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{1-\alpha}} - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (23)$$

where $\alpha < 1$. In this case the energy density and pressures are given by

$$\kappa\rho = \frac{\alpha\left(\frac{r_0}{r}\right)^{3-\alpha}}{r_0^2}, \quad (24)$$

$$\kappa p = -\frac{\left(\frac{r_0}{r}\right)^{3-\alpha}}{r_0^2}, \quad (25)$$

$$\kappa p_l = \frac{(1-\alpha)\left(\frac{r_0}{r}\right)^{3-\alpha}}{2r_0^2}, \quad (26)$$

Note that this spherical spacetime is asymptotically flat and the pressures satisfy linear equations of state. For $0 < \alpha < 1$ the energy density is positive, while for $\alpha < 0$ it becomes negative.

Thus by putting $\phi = 0$ and $b(r) = r_0(r/r_0)^\alpha$ into the field equations (8)-(10) we obtain that the zero-tidal-force hyperbolic counterpart is given by

$$ds^2 = dt^2 - \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{1-\alpha}} - r^2 d\Omega_{ps}^2, \quad (27)$$

$$\kappa\rho = \frac{\alpha\left(\frac{r_0}{r}\right)^{3-\alpha}}{r_0^2} - \frac{2}{r_0^2}\left(\frac{r_0}{r}\right)^2, \quad (28)$$

$$\kappa p_r = -\frac{\left(\frac{r_0}{r}\right)^{3-\alpha}}{r_0^2} + \frac{2}{r_0^2}\left(\frac{r_0}{r}\right)^2, \quad (29)$$

$$\kappa p_l = \frac{(1-\alpha)\left(\frac{r_0}{r}\right)^{3-\alpha}}{2r_0^2}, \quad (30)$$

where $\alpha < 1$. It becomes clear that this zero-tidal-force solution belongs to the hyperbolic M-T wormhole class, while the zero-tidal-force wormhole (15) belongs to the L-M wormhole class.

It can be shown that the energy density (28) is always negative for $r \geq r_0$. The condition $\alpha < 1$ implies that for $r \approx 0$ the first term of Eq. (28) dominates over the second one. Then, $\rho \geq 0$ for $0 < r \leq r_0\left(\frac{2}{\alpha}\right)^{1/(\alpha-1)}$, and $\rho < 0$ for $r > r_0\left(\frac{2}{\alpha}\right)^{1/(\alpha-1)}$. However, for $\alpha < 1$ always $r_0\left(\frac{2}{\alpha}\right)^{1/(\alpha-1)} < r_0$, implying that the energy density is always negative for $r \geq r_0$.

It is interesting to note that the wormholes (23) and (27) satisfy the energy condition $\rho + p_r + 2p_l \geq 0$, since for them we have that $\rho + p_r + 2p_l = 0$.

Both wormholes have the same embedding diagrams for slices $t = const$, $\theta = \pi/2$ in the spherical metric (23), and $t = const$, $\theta = \ln(1 + \sqrt{2})$ in the pseudo-spherical metric (27).

Note that the background on which is constructed the spherical wormhole (23) is the Minkowski spacetime, since for $r_0 = 0$ we have $\rho = p_r = p_l = 0$. On the other hand, the asymptotic spacetime of the metric (23) is also the Minkowski space.

The background on which we construct the hyperbolic wormhole (27) is not an empty spacetime since for the metric

$$ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sinh^2\theta d\phi^2), \quad (31)$$

we have that

$$\kappa\rho = -\kappa p_r = -\frac{2}{r^2}, \quad (32)$$

$$\kappa p_l = 0. \quad (33)$$

This spacetime is not flat, but it is asymptotically locally flat, since for $r \rightarrow \infty$ we obtain that $\rho \rightarrow 0$, $p_r \rightarrow 0$, and for observers, who always remain at rest at constant r , θ and ϕ , we obtain that, at the proper orthonormal basis, the curvature $R_{(\alpha)(\beta)(\gamma)(\delta)} \rightarrow 0$ if $r \rightarrow \infty$.

Thus the interpretation is direct: we have constructed a static wormhole on the non-empty and asymptotically locally flat background (31) by adding the exotic matter source given by (28)-(30) to it. Notice that this is not true for all hyperbolic wormhole spacetimes. See, for example, the wormhole metrics (15) or (36) below: the backgrounds on which these wormholes are constructed and their asymptotic spacetimes do not have the form of the metric (31).

IV. PSEUDO-SPHERICAL WORMHOLES WITH ISOTROPIC PRESSURE

Now, in this section, we shall construct new pseudo-spherical wormholes sustained by a matter content with

isotropic pressure. The main condition for having an isotropic pressure is to require for the radial and lateral pressures the constraint $p_r = p_l$. Thus, for hyperbolic spacetimes, from Eqs. (9) and (10), we have that equation

$$\phi'' + \phi'^2 - \frac{b'r - 3b + 2r}{2r(r-b)} \phi' = \frac{b'r - 3b + 4r}{2r^2(r-b)} \quad (34)$$

must be fulfilled by the metric functions $\phi(r)$ and $b(r)$. From this equation we get for the shape function

$$b(r) = \left(\int \frac{2(r^2 \phi'' + r^2 \phi'^2 - r\phi' - 2) e^{\int \frac{2r^2 \phi'' + 2r^2 \phi'^2 - 3r\phi' - 3}{r(1+r\phi')} dr}}{1 + r\phi'} dr + C \right) e^{-\int \frac{2r^2 \phi'' + 2r^2 \phi'^2 - 3r\phi' - 3}{r(1+r\phi')} dr}, \quad (35)$$

where C is an integration constant. Equations (34) and (35) have a general character since they do not involve an equation of state for the energy density ρ and the isotropic pressure p . Now we have three differential equations for four unknown functions, namely $\phi(r)$, $b(r)$, $\rho(r)$ and $p(r)$. Thus, to study solutions to these field equations, we shall consider restricted choices for one of the unknown functions.

For zero-tidal-force wormhole configurations the restricted form $\phi = \phi_0 = \text{const}$ must be required. By putting $\phi(r) = \text{const}$ into Eq. (35) we obtain a spacetime of constant curvature defined by the shape function $b(r) = 2r + Cr^3$. In this case the metric, the pressure and the energy density may be written in the form

$$ds^2 = dt^2 - \frac{dr^2}{\left(\frac{r}{r_0}\right)^2 - 1} - r^2 (d\theta^2 + \sinh^2 \theta d\phi^2), \quad (36)$$

$p = -\rho/3 = \frac{1}{\kappa r_0^2}$, respectively. Note that the pressure satisfies a barotropic equation of state with a state parameter $\omega = -1/3$, describing a gas of strings. In a

Friedmann-Robertson-Walker universe this gas of strings can change the observable topology of the space [8], while in static spherically symmetric spacetimes its presence produces rather bizarre geometries and it may influence on the rotation curves, mimicking the dark matter effects [8].

The metric (36) describes a wormhole, which belongs to the family of L-M solutions (15). The throat is located at $r = r_0$, the energy density is negative everywhere, and the pressure is always positive. This hyperbolic wormhole is the static version of the evolving wormhole discussed in Ref. [9], which represents a spacetime of two open Friedmann-Robertson-Walker universes connected by a Lorentzian wormhole. The metric (36) is obtained from the evolving wormhole studied in Ref. [9] by making $a(t) = \text{const}$.

Now, we shall construct a master equation for the shape function $b(r)$, by using Eqs. (8), (11) and (34) and assuming the linear equation of state $p_r = p_l = \omega\rho$. The result of these manipulations provides the following equation:

$$\omega(1+\omega)b''' - \frac{2\left(\frac{1}{2} + \omega\right)\omega b''^2}{b' - 2} - \frac{\omega(2r - 6\omega r + (5\omega - 3)b + r(1+\omega)b')b''}{2r(r-b)} +$$

$$\frac{3\left(\frac{4}{3}r(\omega^2 + 1 + 4\omega) - (1 + \frac{16}{3}\omega + \frac{5}{3}\omega^2)b + r(\frac{1}{3} + \omega)(1 + \omega)b'\right)(b' - 2)}{2r^2(r - b)} = 0. \quad (37)$$

It can be shown that the previous discussed solution (36) satisfies the master equation (37) identically as well as $b(r) = 2r - 2\mu$, where μ is a constant. The latter implies that

$$ds^2 = \left(\frac{2\mu}{r} - 1\right) dt^2 - \frac{dr^2}{\frac{2\mu}{r} - 1} - r^2 d\Omega_{ps}^2, \quad (38)$$

with $\rho = p_r = p_l = 0$, and represents the hyperbolic counterpart of the spherically symmetric vacuum Schwarzschild solution. It must be noticed that the metric (38) is static only in the region $0 < r < 2\mu$, therefore this hyperbolic solution does not represent neither a black hole nor a non-traversable wormhole (for a discussion about the spacetime (38) see [7] and the references there in).

Another interesting solution to this master equation is given by $b(r) = 2r - A - \frac{1}{3}Br^3$, and the metric takes the form

$$ds^2 = \left(\frac{A}{r} + \frac{1}{3}Br^2 - 1\right) dt^2 - \frac{dr^2}{\frac{A}{r} + \frac{1}{3}Br^2 - 1} - r^2 d\Omega_{ps}^2,$$

where $p = -\rho = B$. This metric is the hyperbolic version of the spherically symmetric Kottler solution [10]. It is interesting to note that this solution may represent a black hole spacetime for Einstein equations with a cosmological constant [11], and with an appropriate choice of parameter values the spacetime may have a single event horizon.

For $A = 0$ the spacetime (39) becomes a non-traversable hyperbolic wormhole.

A. Hyperbolic wormhole solutions with isotropic pressure for $e^{\phi(r)} = \left(\frac{r}{r_0}\right)^\beta$

Now we shall impose a restricted form for the redshift function. It is of interest to study hyperbolic wormhole solutions generated by the redshift function

$$e^{\phi(r)} = \left(\frac{r}{r_0}\right)^\beta, \quad (40)$$

with β a constant. In the case of spherically symmetric wormholes, such a redshift function generates wormhole spacetimes which are not asymptotically flat, being that in the metric (7) $g_{tt} \rightarrow \infty$ for $r \rightarrow \infty$ and $\beta > 0$, and $g_{tt} \rightarrow 0$ for $r \rightarrow \infty$ and $\beta < 0$.

By putting Eq. (40) into Eq. (34) we obtain

$$b(r) = \frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} r + Cr^{-\frac{(2\beta+1)(\beta-3)}{1+\beta}}, \quad (41)$$

where C is an integration constant. Then the metric is given by

$$ds^2 = \left(\frac{r}{r_0}\right)^{2\beta} dt^2 - \frac{dr^2}{1 - \frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} - C\left(\frac{r}{r_0}\right)^{-\frac{2(\beta^2 - 2\beta - 1)}{1+\beta}}} - r^2 d\Omega_{ps}^2. \quad (42)$$

This metric in the asymptotic limit $r \rightarrow \infty$ describes an hyperbolic spacetime carrying a topological defect for $\frac{(\beta^2 - 2\beta - 1)}{1+\beta} > 0$ and $C > 0$, implying that the parameter β varies in the ranges $-1 < \beta < 1 - \sqrt{2}$ or $1 + \sqrt{2} < \beta$. Thus, if $r \rightarrow \infty$ then $b(r)/r \rightarrow \frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1}$, and the asymptotic spacetime is given by

$$ds^2 = \left(\frac{r}{r_0}\right)^{2\beta} dt^2 - (\beta^2 - 2\beta - 1)dr^2 - r^2 d\Omega_{ps}^2,$$

(39) which, by making $(\beta^2 - 2\beta - 1)r^2 = \varrho^2$, may be rewritten as

$$ds^2 = \left(\frac{\varrho}{\varrho_0}\right)^{2\beta} dt^2 - d\varrho^2 - (\beta^2 - 2\beta - 1)^{-1} \varrho^2 d\Omega_{ps}^2. \quad (43)$$

This metric describes a space with an angle deficit (or excess) and it may be interpreted as an hyperbolic spacetime carrying a topological defect. Such an asymptotic spacetime with a topological defect does not exist for the metric (42) fulfilling the constraints $\frac{(\beta^2 - 2\beta - 1)}{1+\beta} < 0$ and $C < 0$. In this case we obtain that $g_{rr}^{-1} \rightarrow |C| (r/r_0)^{\left|2\frac{(\beta^2 - 2\beta - 1)}{1+\beta}\right|}$ for $r \rightarrow \infty$.

It must be noticed that, we use the term ‘‘angle deficit (or excess)’’, and not the term ‘‘solid angle deficit (or excess)’’, as it is used in the case of spherically symmetric spacetimes, since only the coordinate ϕ describes an angle in the range $[0, 2\pi]$. From the metric (43) we see that for the two-dimensional pseudo-spheres we have that $\alpha^2 \varrho^2 (d\theta^2 + \sinh^2 \theta d\phi^2)$, where $\alpha^2 = (\beta^2 - 2\beta - 1)^{-1} > 0$. Thus, we see that the two-dimensional pseudo-spherical part may be written as $\varrho^2 (d\tilde{\theta}^2 + \sinh^2(\tilde{\theta}/\alpha) d\tilde{\phi}^2)$, where $\tilde{\theta} = \alpha\theta$, $\tilde{\phi} = \alpha\phi$ and $\alpha > 0$, implying that $-\infty < \tilde{\theta} < \infty$ and $0 \leq \tilde{\phi} \leq 2\pi\alpha$. It becomes clear that the new angle $\tilde{\phi}$ has an angle deficit for $0 < \alpha < 1$, and an angle excess for $\alpha > 1$.

Now, we shall show that the solution (42) includes hyperbolic L-M wormholes, as well as hyperbolic M-T ones.

Metric	Constraints	Class	$\rho \geq 0$	$\rho + p_i \geq 0$	$\rho + p_r + 2p_l \geq 0$
$ds^2 = dt^2 - \frac{dr^2}{\left(\frac{r}{r_0}\right)^{\frac{1+\omega}{\omega}} - 1} - r^2 d\Omega_{ps}^2,$	$p_r = \omega\rho, \Phi(r) = 0,$ $-1 < \omega < 0$	L-M	No	No	No
$ds^2 = \left(\frac{r}{r_0}\right)^{-2(\omega+1)} dt^2 - \frac{dr^2}{1 - \left(\frac{r}{r_0}\right)^{\frac{1+\omega}{\omega}}} - r^2 d\Omega_{ps}^2,$	$p_r = \omega\rho,$ $-1 < \omega < 0$	M-T	No	No	No
$ds^2 = dt^2 - \frac{dr^2}{1 - \left(\frac{r}{r_0}\right)^{1-\alpha}} - r^2 d\Omega_{ps}^2,$	$\Phi(r) = 0,$ $\alpha < 1$	M-T	No	No	Yes
$ds^2 = dt^2 - \frac{dr^2}{\left(\frac{r}{r_0}\right)^2 - 1} - r^2 d\Omega_{ps}^2$	$p_r = p_l, \Phi(r) = 0$	L-M	No	No	No
$ds^2 = \left(\frac{r}{r_0}\right)^{2(1+\sqrt{3})} dt^2 - \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{2/(2+\sqrt{3})}} - r^2 d\Omega_{ps}^2$	$p_r = p_l, e^{\Phi(r)} = \left(\frac{r}{r_0}\right)^{1+\sqrt{3}}$	M-T	No	No	Yes
$ds^2 = \left(\frac{r}{r_0}\right)^{2(1-\sqrt{3})} dt^2 - \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{2/(2-\sqrt{3})}} - r^2 d\Omega_{ps}^2$	$p_r = p_l, e^{\Phi(r)} = \left(\frac{r}{r_0}\right)^{1-\sqrt{3}}$	M-T	No	No	No
$ds^2 = \left(\frac{r}{r_0}\right)^4 dt^2 - \frac{dr^2}{\left(\frac{r}{r_0}\right)^{2/3} - 1} - r^2 d\Omega_{ps}^2$	$p_r = p_l, e^{\Phi(r)} = \left(\frac{r}{r_0}\right)^2$	L-M	No	No	Yes
$ds^2 = \left(\frac{e}{e_0}\right)^{2\beta} dt^2 - \frac{de^2}{\left(\frac{e}{e_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}} - 1} - \left(\frac{\beta^2-2\beta-2}{\beta^2-2\beta-1} - 1\right) e^2 d\Omega_{ps}^2,$	$p_r = p_l, e^{\Phi(r)} = \left(\frac{r}{r_0}\right)^\beta,$ $0 < \beta < 2$	L-M	No	No	Yes
$ds^2 = \left(\frac{e}{e_0}\right)^{2\beta} dt^2 - \frac{de^2}{\left(\frac{e}{e_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}} - 1} - \left(\frac{\beta^2-2\beta-2}{\beta^2-2\beta-1} - 1\right) e^2 d\Omega_{ps}^2,$	$p_r = p_l, e^{\Phi(r)} = \left(\frac{r}{r_0}\right)^\beta,$ $1 - \sqrt{2} < \beta < 0$ or $2 < \beta < 1 + \sqrt{2}$	L-M	No	No	Yes
$ds^2 = \left(\frac{e}{e_0}\right)^{2\beta} dt^2 - \frac{de^2}{1 - \left(\frac{e}{e_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}} - \left(1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1}\right) e^2 d\Omega_{ps}^2,$	$p_r = p_l, e^{\Phi(r)} = \left(\frac{r}{r_0}\right)^\beta,$ $\beta < 1 - \sqrt{3}$ or $1 + \sqrt{3} < \beta$	M-T	No	No	Yes
$ds^2 = \left(\frac{e}{e_0}\right)^{2\beta} dt^2 - \frac{de^2}{1 - \left(\frac{e}{e_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}} - \left(1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1}\right) e^2 d\Omega_{ps}^2,$	$p_r = p_l, e^{\Phi(r)} = \left(\frac{r}{r_0}\right)^\beta,$ $1 + \sqrt{2} < \beta < 1 + \sqrt{3}$ or $1 - \sqrt{3} < \beta < 1 - \sqrt{2}$	M-T	No	No	Yes

TABLE I: In this table all discussed by us wormhole solutions are listed. We indicate to which class a given solution belongs and which of energy conditions $\rho + p_i \geq 0$, $\rho + p_i \geq 0$ and $\rho + p_r + 2p_l \geq 0$ are satisfied or not. We can see that there are wormholes of L-M and M-T class which satisfy the energy condition $\rho + p_r + 2p_l \geq 0$. Notice that for wormholes with isotropic pressure the condition $\rho + p_r + 2p_l \geq 0$ becomes $\rho + 3p \geq 0$.

1. Wormholes without angle deficit or excess

As we have stated above the solution (42) is a space-time carrying a topological defect. We are interested first in considering pseudo-spherical wormhole geometries without such a topological defect. Hence we must require for the metric (42) that

$$1 - \frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} = \pm 1. \quad (44)$$

Therefore, we have $\beta = 1 \pm \sqrt{3}$ for the upper sign, and $\beta = 0, 2$ for the lower sign.

Let us now consider the values $\beta = 1 \pm \sqrt{3}$. In this case the solutions are given by

$$ds^2 = \left(\frac{r}{r_0}\right)^{2(1\pm\sqrt{3})} dt^2 - \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{2/(2\pm\sqrt{3})}} - r^2 d\Omega_{ps}^2, \quad (45)$$

$$\kappa\rho = \frac{r_0^4(\pm 2\sqrt{3} - 3)}{r^6 \left(\frac{r_0}{r}\right)^{\pm 2\sqrt{3}}} - \frac{2}{r^2}, \quad (46)$$

$$\kappa p = \frac{r_0^4(\mp 2\sqrt{3} - 3)}{r^6 \left(\frac{r_0}{r}\right)^{\pm 2\sqrt{3}}} + \frac{2(2 \pm \sqrt{3})}{r^2}, \quad (47)$$

where we have put, without any loss of generality, $C = 1$. Note that Eq. (45) implies that $g_{rr} \rightarrow 1$ if $r \rightarrow \infty$, and then the M-T constraint 3 is satisfied. Also these solutions satisfy the flare-out condition $\frac{b-b'r}{2b^2} > 0$, which in this case is clearly satisfied since

$$\frac{b - b'r}{2b^2} = \frac{\left(\frac{r_0}{r}\right)^{-\frac{2}{2\pm\sqrt{3}}}}{r(2 \pm \sqrt{3})} > 0, \quad (48)$$

for any $r \geq r_0$.

On the other hand, the case $\beta = 0$ was already discussed (see Eq. (36)). For $\beta = 2$ we obtain the solution

$$\begin{aligned} ds^2 &= \left(\frac{r}{r_0}\right)^4 dt^2 - \frac{dr^2}{\left(\frac{r}{r_0}\right)^{2/3} - 1} - r^2 d\Omega_{ps}^2, \\ \kappa\rho &= -\frac{5}{3r_0^2} \left(\frac{r_0}{r}\right)^{4/3}, \\ \kappa p &= \frac{5}{r_0^2} \left(\frac{r_0}{r}\right)^{4/3} - \frac{4}{r^2}. \end{aligned} \quad (49)$$

where we have put $C = -1$ in order to have the right signature in the metric.

In conclusion these four solutions without a topological defect belong to two different classes of wormholes: solutions (36) and (49) belong to the hyperbolic class of wormholes discussed by Lobo and Mimoso in Ref. [7], and

they clearly do not satisfy the M-T constraint 3, while the spacetimes (45) belong to the M-T wormhole class.

2. Hyperbolic L-M wormholes with an angle deficit or excess

We now provide a physical/geometric interpretation of the obtained pseudo-spherical spacetime (42), by considering the presence of a topological defect in the metric (42) for any $r \geq r_0$. It can be shown that this metric has space sections with an angle deficit (or excess) in the case of L-M wormhole class as well as of M-T one.

For having a wormhole configuration we must require $b(r_0) = r_0$. Then, the metric (42), energy density and pressure are provided by the expressions

$$ds^2 = \left(\frac{r}{r_0}\right)^{2\beta} dt^2 - \frac{dr^2}{\left(\frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} - 1\right) \left(\left(\frac{r}{r_0}\right)^{-\frac{2(\beta^2 - 2\beta - 1)}{1+\beta}} - 1\right)} - r^2 d\Omega_{ps}^2, \quad (50)$$

$$\kappa\rho(r) = -\frac{(2\beta + 1)(\beta - 3)}{(1 + \beta)(\beta^2 - 2\beta - 1)r_0^2} \left(\frac{r_0}{r}\right)^{\frac{2\beta(\beta - 1)}{1+\beta}} - \frac{(\beta - 2)\beta}{(-2\beta - 1 + \beta^2)r^2}, \quad (51)$$

$$\kappa p(r) = -\frac{2\beta + 1}{(\beta^2 - 2\beta - 1)r_0^2} \left(\frac{r_0}{r}\right)^{\frac{2\beta(\beta - 1)}{1+\beta}} + \frac{\beta^2}{(\beta^2 - 2\beta - 1)r^2}, \quad (52)$$

respectively. It becomes clear from Eqs. (51) and (52) that the fluid density and the isotropic pressure are not related by a linear equation of state.

We have such a linear equation of state only for $\beta = 0$ and $\beta = -1/2$. For vanishing β we obtain $\rho/3 = -p = -\frac{1}{\kappa r_0^2}$, i.e. $\omega = -1/3$. This solution is the discussed above zero-tidal-force L-M wormhole (36). For $\beta = -1/2$ we obtain for the metric, energy density and pressure that

$$ds^2 = \frac{r_0}{r} dt^2 - \frac{dr^2}{4\left(1 - \frac{r_0}{r}\right)} - r^2 d\Omega_{ps}^2 \quad (53)$$

$$\kappa\rho = -\frac{5}{r^2}, \quad (54)$$

$$\kappa p = \frac{1}{r^2}. \quad (55)$$

As we will discuss below, such a metric represent a M-T hyperbolic wormhole exhibiting an angle excess.

Let us first show that the metric (42) includes, as particular solutions, pseudo-spherical spacetimes of L-M wormhole class carrying a topological defect. The metric (50) describes L-M pseudo-spherical wormholes if $\frac{(\beta^2 - 2\beta - 1)}{1+\beta} < 0$, i.e. for $1 - \sqrt{2} < \beta < 1 + \sqrt{2}$. Note that at this range $\frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} - 1 > 0$. By making, for the

radial coordinate, the rescaling

$$r = \pm \sqrt{\frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} - 1} \varrho, \quad (56)$$

the metric (50) takes the form

$$ds^2 = \left(\frac{\varrho}{\varrho_0}\right)^{2\beta} dt^2 - \frac{d\varrho^2}{\left(\frac{\varrho}{\varrho_0}\right)^{-\frac{2(\beta^2 - 2\beta - 1)}{1+\beta}} - 1} - \left(\frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} - 1\right) \varrho^2 d\Omega_{ps}^2. \quad (57)$$

The condition $-\frac{1}{\beta^2 - 2\beta - 1} = \frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} - 1 > 0$ ensures the right signature of the metric (57). For $1 < \frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} < 2$, i.e. for $0 < \beta < 2$, we have a spacetime with an angle deficit, and for $\frac{\beta^2 - 2\beta - 2}{\beta^2 - 2\beta - 1} > 2$, i.e. for $1 - \sqrt{2} < \beta < 0$ or $2 < \beta < 1 + \sqrt{2}$, we have a spacetime with an angle excess. Notice that the power of ϱ/ϱ_0 in the radial metric component of Eq. (57) is positive for $0 < \beta < 2$, so there exist wormhole of L-M class with an angle deficit. For $1 - \sqrt{2} < \beta < 0$ and $2 < \beta < 1 + \sqrt{2}$ the power $-\frac{2(\beta^2 - 2\beta - 1)}{1+\beta}$ is also positive, so also there exist wormholes of the L-M

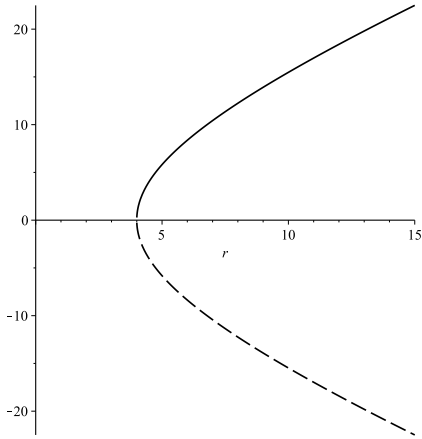


FIG. 1: The figure shows the embedding diagram of the wormhole (66) obtained from Eq. (65) for the parameter values $\beta = 3, r_0 = 4$, where the throat is located. The solid line correspond to + sign and the dashed line to - sign in Eq. (65). The embedding shows that the wormhole extends from the throat at $r = 4$ to infinity. For a full visualization of the surface the diagram must be rotated about the vertical z axis (see Fig. (3)).

class with an angle excess. In this case the energy density and pressure are given by

$$\begin{aligned} \kappa\rho(\varrho) &= \frac{(2\beta+1)(\beta-3)}{(1+\beta)\varrho_0^2} \left(\frac{\varrho}{\varrho_0}\right)^{-\frac{2\beta(\beta-1)}{1+\beta}} + \frac{(\beta-2)\beta}{\varrho^2}, \\ \kappa p(\varrho) &= \frac{(2\beta+1)}{\varrho_0^2} \left(\frac{\varrho}{\varrho_0}\right)^{-\frac{2\beta(\beta-1)}{1+\beta}} - \frac{\beta^2}{\varrho^2}. \end{aligned} \quad (58)$$

It is interesting to note, that for $\frac{\beta^2-2\beta-1}{1+\beta} < 0$, the wormhole condition $b(r_0) = r_0$ has induced for the metric (50) the behavior $g_{rr}^{-1} \rightarrow \frac{1}{1+2\beta-\beta^2} (r/r_0)^{\left|2\frac{(\beta^2-2\beta-1)}{1+\beta}\right|}$ for $r \rightarrow \infty$, i.e. the asymptotic spacetime exhibits an angle deficit (or excess).

3. Hyperbolic M-T wormholes with an angle deficit or excess

In order to metric (50) describes solutions of the M-T wormhole class the M-T Constraint 3 must be satisfied. Hence we must require that $\frac{\beta^2-2\beta-1}{1+\beta} > 0$, implying that the parameter β varies in the ranges $-1 < \beta < 1 - \sqrt{2}$ or $1 + \sqrt{2} < \beta$. In this case the metric (50) may be rewritten in the equivalent form

$$ds^2 = \left(\frac{r}{r_0}\right)^{2\beta} dt^2 - \frac{dr^2}{\left(1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1}\right) \left(1 - \left(\frac{r}{r_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}\right)} - r^2 d\Omega_{ps}^2. \quad (59)$$

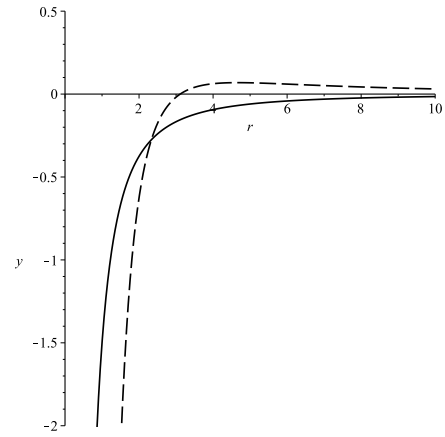


FIG. 2: The figure shows the behavior of the energy density (solid line) and the pressure (dashed line). The energy density is always negative, while the pressure is always positive for $r \geq 4$ (note that $p=0$ at $r = 28/9 < 4$, where the throat is located).

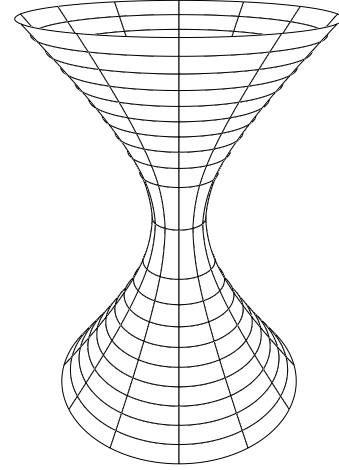


FIG. 3: The figure shows the three-dimensional embedding diagram for the wormhole of Fig. (1).

If $r \rightarrow \infty$ we obtain $g_{rr}^{-1} \rightarrow 1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1}$, implying that the asymptotic spacetime is given by the metric (43), which describes a space with an angle deficit (or excess). By making, for the radial coordinate, the rescaling

$$r = \pm \sqrt{1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1}} \varrho, \quad (60)$$

the metric (50) takes the form

$$ds^2 = \left(\frac{\varrho}{\varrho_0}\right)^{2\beta} dt^2 - \frac{d\varrho^2}{1 - \left(\frac{\varrho}{\varrho_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}} - \left(1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1}\right) \varrho^2 d\Omega_{ps}^2. \quad (61)$$

The condition $\frac{1}{\beta^2-2\beta-1} = 1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1} > 0$ must be

required in order to ensure the right signature of the metric (61). For $0 < \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1} < 1$, i.e. for $\beta > 1 + \sqrt{3}$ or $\beta < 1 - \sqrt{3}$, we have a spacetime with an angle deficit, and for $\frac{\beta^2-2\beta-2}{\beta^2-2\beta-1} < 0$, i.e. for $1 + \sqrt{2} < \beta < 1 + \sqrt{3}$ or $1 - \sqrt{3} < \beta < 1 - \sqrt{2}$, we have a spacetime with an angle excess. Notice that the power of ϱ/ϱ_0 in the radial metric component of Eq. (61) is negative for $\beta > 1 + \sqrt{3}$ and $-1 < \beta < 1 - \sqrt{3}$, so there exist wormhole of M-T class with an angle deficit. For $1 + \sqrt{2} < \beta < 1 + \sqrt{3}$ and $1 - \sqrt{3} < \beta < 1 - \sqrt{2}$ the power $-\frac{2(\beta^2-2\beta-1)}{1+\beta}$ is also negative, so also there exist wormholes of the M-T class with an angle excess. In this case the energy density and pressure are given by

$$\begin{aligned}\kappa\rho(\varrho) &= -\frac{(2\beta+1)(\beta-3)}{(1+\beta)\varrho_0^2} \left(\frac{\varrho}{\varrho_0}\right)^{-\frac{2\beta(\beta-1)}{1+\beta}} - \frac{(\beta-2)\beta}{\varrho^2}, \\ \kappa p(\varrho) &= -\frac{(2\beta+1)}{\varrho_0^2} \left(\frac{\varrho}{\varrho_0}\right)^{-\frac{2\beta(\beta-1)}{1+\beta}} + \frac{\beta^2}{\varrho^2}.\end{aligned}\quad (62)$$

We note that expressions (58) and (62) for the energy density and isotropic pressures are identical up to the sign.

Now, in order for the spacetime (50) to see the shape of a wormhole of the hyperbolic M-T class, we will embed space slices of the metric (50), or equivalently of the metric (59), satisfying the conditions $\frac{\beta^2-2\beta-1}{1+\beta} > 0$ and $1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1} > 0$, as surfaces of revolution in an Euclidean 3-dimensional space. For producing embeddings we shall consider two dimensional space slices of the metric (50) by making the restriction $t = t_0 = \text{const}$ and $\sinh(\theta_0) = 1$ (which implies that $\theta_0 = \ln(1 \pm \sqrt{2})$). With these constraints the respective two-dimensional line element may be written as

$$ds^2 = \frac{dr^2}{\left(1 - \frac{\beta^2-2\beta-2}{\beta^2-2\beta-1}\right) \left(1 - \left(\frac{r}{r_0}\right)^{-\frac{2(\beta^2-2\beta-1)}{1+\beta}}\right)} + r^2 d\phi^2,\quad (63)$$

To visualize this space slice we identify Eq. (63) as the metric of a surface of revolution in R^3 , embedded into Euclidean metric, written in cylindrical coordinates as

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2.\quad (64)$$

Comparing both metrics we conclude that

$$\frac{dz}{dr} = \pm \frac{1}{\sqrt{\frac{\beta^2-2\beta-1}{\beta^2-2\beta-2 + \left(\frac{r_0}{r}\right)^{\frac{2(\beta^2-2\beta-1)}{1+\beta}}} - 1}}.\quad (65)$$

Let us briefly discuss particular solutions defined by the parameter values $\beta = 3, r_0 = 4$ and $\beta = -1/2, r_0 = 4$.

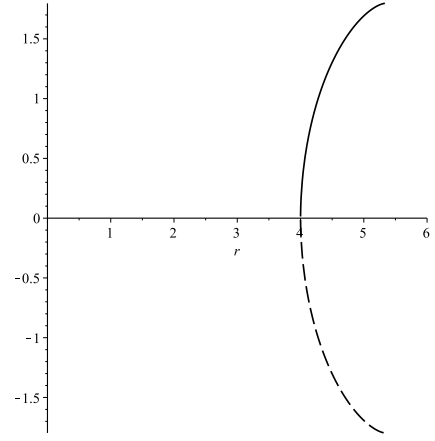


FIG. 4: The figure shows the embedding diagram of the wormhole (69) obtained from Eq. (65) for the parameter values $\beta = -1/2, r_0 = 4$, where the throat is located. The solid line correspond to + sign and the dashed line to - sign in Eq. (65). In this case the wormhole extends from the throat, at $r = 4$, to $r = 16/3$. For a full visualization of the surface the diagram must be rotated about the vertical z axis (see Fig. (5)).

For $\beta = 3$ and $r_0 = 4$ the respective metric, energy density and pressure are given by

$$ds^2 = \frac{r^2}{64} dt^2 - \frac{dr^2}{\frac{1}{2} - \frac{2}{r}} - r^2 d\Omega_{ps}^2,\quad (66)$$

$$\kappa\rho = -\frac{3}{2r^2},\quad (67)$$

$$\kappa p = -\frac{14}{r^3} + \frac{9}{2r^2}.\quad (68)$$

This spacetime describes an hyperbolic wormhole with a throat located at $r = 4$, and in the asymptotic limit we obtain a spacetime with an angle deficit. In this case, the coordinates θ and ϕ are redefined as follows:

$$-\infty < \tilde{\theta} = \theta/\sqrt{2} < \infty \text{ and } 0 \leq \tilde{\phi} = \phi/\sqrt{2} \leq \sqrt{2}\pi.$$

In Fig. (1) we show the embedding of this hyperbolic wormhole, while in Fig. (2) the behavior of the energy density and the isotropic pressure are shown.

Now, for $\beta = -1/2$ and $r_0 = 4$ the metric is provided by

$$ds^2 = \frac{1}{r} dt^2 - \frac{dr^2}{4 - \frac{16}{r}} - r^2 d\Omega_{ps}^2,\quad (69)$$

while the energy density and pressure are given by Eqs. (54) and (55) respectively. This spacetime describes an hyperbolic wormhole with a throat located at $r = 4$, and in the asymptotic limit we obtain a spacetime with an angle excess. The coordinates θ and ϕ are redefined as follows: $-\infty < \tilde{\theta} = 2\theta < \infty$ and $0 \leq \tilde{\phi} = 2\phi \leq 4\pi$. The embedding shows (see Fig. (4)) that the wormhole extends from the throat, at $r = 4$, to $r = 16/3$.

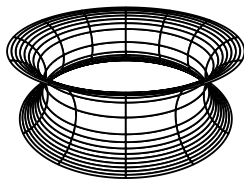


FIG. 5: The figure shows the three-dimensional embedding diagram for the wormhole of Fig. (4).

V. CONCLUSIONS

We have shown that in pseudo-spherically symmetric spacetimes there are two classes of static wormhole solutions: one of them discussed by Lobo and Mimoso in Ref. [7], and another one discussed by us in this paper. The L-M class may be written in the form (5) and is characterized by the condition (6), while the second class of hyperbolic wormholes may be written in the form (7) and is characterized by the M-T condition (2). The use of Eqs. (2), (5), (6) and (7) is the main criterion to determine to which class of wormholes a given hyperbolic solution belongs.

We obtain exact solutions belonging to both classes of static and pseudo-spherically symmetric spacetime wormholes. The specific studied solutions are obtained by considering several equations of state or by imposing restricted choices for the redshift function and/or the shape function.

It is interesting to remark that for all obtained pseudo-spherical wormhole solutions we have that at the throat the energy density is negative, while the radial pressure positive. In general, it can be shown that this is true for any hyperbolic wormhole, including L-M wormholes, as well as M-T ones. Effectively, from the radial metric component in metric (7) we may write for the proper radial distance that

$$l(r) = \pm \int_{r_0}^r \frac{dr'}{\sqrt{1 - \frac{b(r')}{r'}}}. \quad (70)$$

From this expression it can be shown that at the throat always the relation

$$b'(r_0) \leq 1 \quad (71)$$

is satisfied [2]. Thus, from Eq. (8) we obtain for the energy density $\kappa\rho(r_0) \leq -1/r_0^2$, implying that at the throat the energy density is always negative.

On the other hand, if the redshift function is finite for $r \geq r_0$, Eq. (9) implies that $\kappa p_r(r_0) = 1/r_0^2$, so the radial pressure is always positive at the throat. Notice that $\kappa\rho + \kappa p_r|_{r_0} = (b'(r_0) - 1)/r_0^2$. Thus, by taking into account Eq. (71) we conclude that at the throat $\rho + p_r|_{r_0} \leq 0$.

From Eq. (10) we obtain that at the throat the relation $\kappa p_l(r_0) = (1 - b')(1 + \phi'(r_0)r_0 + 1)/(2r_0^2)$ is valid for the lateral pressure. Therefore, Eq. (71) implies that, at the throat, for $\phi'(r_0) > -1/r_0$ the lateral pressure is

positive, while for $\phi'(r_0) < -1/r_0$, the lateral pressure is negative. This is why for the zero-tidal-force hyperbolic wormholes (15) and (27) the lateral pressures (17) and (30) are positive. In conclusion, it follows from these results that an hyperbolic wormhole can not be sustained at the throat by phantom energy, since always the energy density is negative, and at least the radial pressure is always positive.

Notice that in opinion of the authors of [7] the static hyperbolic tunnels are constructed by adding exotic matter to the vacuum solution (38), which is the pseudo-spherical vacuum counterpart of the Schwarzschild solution.

On the light of our results, this statement is not true, since, strictly speaking, the metric (38) should be obtained from any hyperbolic wormhole when the exotic matter vanishes (we should obtain a background solution with $g_{tt} = 2\mu/r - 1$, which vanishes at $r = 2\mu$). This clearly does not happen, even in the case of hyperbolic L-M wormholes discussed in Ref. [7].

In our case, most of static hyperbolic M-T wormholes are constructed by adding exotic matter to the spacetime (31). This spacetime is the hyperbolic counterpart of the Minkowski background (and not of the vacuum Schwarzschild solution), on which most of spherically symmetric wormholes are constructed by adding exotic matter to it.

For example, for the zero-tidal-force hyperbolic M-T wormhole (27) we have that if $r_0 = 0$ the background solution (31) is obtained. While, for the zero-tidal-force hyperbolic L-M wormhole (15), neither the metric (31) nor (38) are the background spacetimes on which the solution (15) is constructed by adding exotic matter to one of them.

In the table I all discussed by us wormhole solutions are listed. We indicate to which class a given solution belongs and which energy conditions are satisfied.

Lastly, let us remark that the counterpart of the static and spherical self-dual Lorentzian wormhole discussed in Ref. [12] is not a wormhole in pseudo-spherically symmetric spacetimes. Effectively, this two-parameter family of spherically symmetric, static Lorentzian wormholes is obtained as the general solution of the equations $\rho(r) = 0$ and $T_{ij} - \frac{1}{2}Tg_{ij}u^i u^j = 0$, where $u^i u_j = 1$. All these solutions have a vanishing scalar curvature $R = 0$. The conditions required for these self-dual Lorentzian wormholes imply that in pseudo-spherical spacetimes the metric is given by

$$ds^2 = \left(k + \lambda \sqrt{\frac{2m}{r} - 1} \right)^2 dt^2 - \frac{dr^2}{\frac{2m}{r} - 1} - r^2 d\Omega_{ps}^2,$$

where k , λ and m are constants of integration. This class of solutions, as well as the spherical symmetric one, includes the vacuum hyperbolic version of the spherically symmetric Schwarzschild solution, which is obtained by requiring $k = 0$. This solution is obtained by adding a matter source to the vacuum metric (38), however, in

order to have $g_{rr} > 0$ we must require that $0 < r < 2m$, and then this spacetime does not describe a wormhole geometry.

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- [1] M. S. Morris and K. S. Thorne, *Am. J. Phys.* **56**, 395 (1988); M. S. Morris, K. S. Thorne and U. Yurtsever, *Phys. Rev. Lett.* **61**, 1446 (1988).
- [2] M. Visser, *Lorentzian Wormholes: From Einstein to Hawking*, (American Institute of Physics, New York, 1995).
- [3] J. P. S. Lemos and F. S. N. Lobo, *Phys. Rev. D* **69**, 104007 (2004) [gr-qc/0402099]; C. Barcelo, L. J. Garay, P. F. Gonzalez-Diaz and G. A. Mena Marugan, *Phys. Rev. D* **53**, 3162 (1996); J. P. S. Lemos, F. S. N. Lobo and S.Q. de Oliveira, *Phys. Rev. D* **68**, 064004 (2003) [gr-qc/0302049].
- [4] S. V. Sushkov, *Phys. Rev. D* **71**, 043520 (2005) [gr-qc/0502084];
- [5] F. S. N. Lobo, *Phys. Rev. D* **71**, 084011 (2005).
- [6] M. R. Mehdizadeh, M. K. Zangeneh and F. S. N. Lobo, *Phys. Rev. D* **91**, no. 8, 084004 (2015) [arXiv:1501.04773 [gr-qc]]; H. Saiedi, *Mod. Phys. Lett. A* **27**, 1250220 (2012) [arXiv:1409.2179 [gr-qc]]; P. Kanti, B. Kleihaus and J. Kunz, *Phys. Rev. Lett.* **107**, 271101 (2011) [arXiv:1108.3003 [gr-qc]]; H. Maeda and M. Nozawa, *Phys. Rev. D* **78**, 024005 (2008) [arXiv:0803.1704 [gr-qc]]; J. Matulich and R. Troncoso, *JHEP* **1110**, 118 (2011) [arXiv:1107.5568 [hep-th]]; S. H. Mazharimousavi, M. Halilsoy and Z. Amirabi, *Phys. Rev. D* **81**, 104002 (2010) [arXiv:1001.4384 [gr-qc]]; M. H. Dehghani and M. R. Mehdizadeh, *Phys. Rev. D* **85**, 024024 (2012) [arXiv:1111.5680 [gr-qc]]; M. G. Richarte and C. Simone, *Phys. Rev. D* **76**, 087502 (2007); *Phys. Rev. D* **77**, 089903 (2008) [arXiv:0710.2041 [gr-qc]]; M. G. Richarte, *Phys. Rev. D* **82**, 044021 (2010) [arXiv:1001.4034 [gr-qc]].
- [7] F. S. N. Lobo and J. P. Mimoso, *Phys. Rev. D* **82**, 044034 (2010) [arXiv:0907.3811 [gr-qc]].
- [8] A. Y. Kamenshchik and I. M. Khalatnikov, *Int. J. Mod. Phys. D* **21**, 1250004 (2012) [arXiv:1109.0201 [gr-qc]].
- [9] L. X. Li, *J. Geom. Phys.* **40**, 154 (2001).
- [10] F. Kottler, *Annalen der Physik* **56** 401 (1918).
- [11] R. G. Cai, J. Y. Ji and K. S. Soh, *Phys. Rev. D* **57**, 6547 (1998); R. G. Cai and A. Wang, *Phys. Rev. D* **70**, 064013 (2004)
- [12] N. Dadhich, S. Kar, S. Mukherjee and M. Visser, *Phys. Rev. D* **65**, 064004 (2002) [gr-qc/0109069].